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Chromatically Optimal Rigid Graphs

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The purpose of this paper is to prove that a given graph G with chromatic number k (k finite or infinite) is a subgraph of a k-chromatic rigid graph H if and only if G does not contain a complete graph with k vertices. This solves a problem of L. Babai and J. Nešetřil.

INTRODUCTION

A rigid graph is a graph G for which the identical mapping is the unique homomorphism $G \to G$.

Recall that a map f: $V(G) \to V(H)$ is called a homomorphism $G \to H$ if $\{f(x), f(y)\} \in E(G)$ for every edge $\{x, y\} \in E(G)$.

Rigid graps were defined in [6] and studied, e.g., in [1 – 3, 6, 8, 9, 11]. Several papers were devoted to the study of subgraphs of rigid graphs [3, 8, 1, 2].

Particularly, extending an earlier partial result, [1] proves that every graph is an induced subgraph of a rigid graph. Reference [2] presents another proof of this fact.

All these constructions have the common property that they extend a given graph to a rigid graph with an increase of chromatic number.

This is of course necessary for those k-chromatic graphs which contain K_k—the complete graph with k vertices—as a subgraph.

We prove here that this exception is the unique exception. This statement is stated below as Theorem 2.1. Theorem 2.2 presents a strengthening of this result for finite graphs by proving that both the

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chromatic number and the girth may coincide for every graph and its rigid extension.

Theorem 2.1 solves a problem found in [2].

In Section 3 we shall further strengthen the properties of rigid graphs and we state some negative results.

1. Uncomparable Sparse Graphs with Given Chromatic Number

The purpose of this section is to prove two statements which are instrumental to our proof of the main result:

PROPOSITION 1.1. *Let l be a positive integer and let k be (finite or infinite) cardinal. Let G be a graph not containing K_k which has chromatic number k. Then there exists a graph H with the following properties:*

1. $\chi(G) = \chi(H) = k$,
2. *H has no circuit of an odd length less than l,*
3. *$H \nrightarrow G$ (i.e., there is no homomorphism from H to G).*

PROPOSITION 1.2. *Let k, l be positive integers. Let G be a finite graph not containing K_k which has chromatic number k. Then there exists a graph H with the following properties:*

1. $\chi(G) = \chi(H) = k$,
2. *H has girth $\geqslant l$,*
3. *$H \nrightarrow G$ (i.e., there is no homomorphism from H to G).*

Note that Proposition 1.2 does not have an infinite analogue as every graph with $\chi(G) = \aleph_1$ contains a rectangle. We prove Proposition 1.2 by probabilistic means. M. Fiklíková (Prague) recently found a constructive proof of Proposition 1.2.

Proof of Proposition 1.1. Let G, l, k be fixed. Put $|V(G)| = \lambda$. Let G' be a connected graph with the following properties:

1. $\chi(G') > \lambda^k$,
2. G' does not contain an odd cycle of length $< l$. For the existence G' see, e.g., [5].

Put $H = G' \times K_k$ where $\times$ is the direct product: $\{(x, \alpha), (y, \beta)\}$ is an edge of H iff $\alpha \neq \beta$ and $\{x, y\} \in E(G')$.

We shall verify properties 1, 2, 3 of Proposition 1.1. Clearly it suffices to consider 3.

In the way of contradiction let there exists a homomorphism f: $H \to G$. Define a mapping g: $V(G') \to \lambda^k$ by $g(x) = (f(x, \alpha)/\alpha < k)$.

As g fails to be a colouring there exists an edge $\{x, y\}$ of G' such that $g(x)=g(y)$ and thus $f(x, \alpha)=g(y, \alpha)$ for every $\alpha<k$. This in turn means that the homomorphic image $f[H]$ contains K_k which is a contradiction. ▮

Proof of Proposition 1.2. Let k, l, and G be given. Put $r=|V(G)|$. For a positive integer n consider pairwise disjoint sets $A_1, A_2, ..., A_k$ each of size n.

Let $\mathbf{G}$ be a random graph with the vertex set $V=\bigcup_{i=1}^{k} A_i$ where the edges are chosen independently from the set $\binom{V}{2}-\bigcup_{i=1}^{k}\binom{A_i}{2}$ each with the probability $p=n^{\delta-1}$, where $0<\delta<1/l$. (We use $\binom{V}{2}$ so denote the set of two-elements subsets of V).

It will be convenient to introduce the following notion: we say that the set $B\subseteq V$ is *large* if there are i, j, $1\leqslant i<j\leqslant k$, such that $|B\cap A_i|\geqslant n/r$ and $|B\cap A_j|\geqslant n/r$.

First we bound

$$\alpha=\operatorname{Prob}[B \text{ is large implies } |\mathbf{G}/B|\geqslant n]$$

($\mathbf{G}/B$ denotes the set of edges of $\mathbf{G}$ which lie in B). We have

$$1-\alpha\leqslant \sum_{B \text{ large}} \operatorname{Prob}[|\mathbf{G}/B|<n]\leqslant 2^{kn}\binom{\binom{kn}{2}}{n}(1-p)^{n^2/r^2}$$

$$<n^{c_1 n\log n-c_2 n^{1+\delta}}=o(1),$$

where c_1, c_2 are positive constants independent on n.

Thus

$$\operatorname{Prob}[B \text{ is large implies } |\mathbf{G}/B|\geqslant n]=1-o(1). \tag{1}$$

On the other hand if we denote by $c(\mathbf{G})$ the number of edges contained in all cycles of length 3, ..., l we have the expected value

$$E(c(G))\leqslant 3!\binom{kn}{3}p^3+4!\binom{kn}{4}p^4+\cdots+l!\binom{kn}{l}p^l=o(n). \tag{2}$$

Combining (1) and (2) we get that if n is large enough there exists an instance H^* of $\mathbf{G}$ such that

(i) $|\binom{B}{2}\cap E(H^*)|\geqslant n$ for any large subset $B\subset V$;

(ii) there exist $n-1$ edges $e_1, ..., e_{n-1}$ such that the girth $H\geqslant l$ for the graph H, $V(H)=V(H^*)$, $E(H)=E(H^*)-\{e_1, ..., e_{n-1}\}$.

We prove that the graph H satisfies properties 1, 2, 3. Properties 1 and 2 are evident.

To prove 3 assume on the contrary that there exists a homomorphism $f\colon H \to G$. From (i) and (ii) follows that none of the sets $f^{-1}(x)$, $x \in V(G)$, is large. Hence there are distinct $x_1, ..., x_k$ vertices of G such that $|f^{-1}(x_i) \cap A_i| \geqslant n/r$ for all $i = 1, ..., k$. Using (i) and (ii) again we get that $x_1, ..., x_k$ span a complete graph in G. This is a contradiction. ∎

2. Rigid Graphs

We prove the results announced in the Introduction.

THEOREM 2.1. *For every (possibly infinite) k and for every graph G with $\chi(G) = k$, $G \not\supseteq K_k$, there exists a graph H with properties*

1. $\chi(H) = k$,
2. *G is an induced subgraph of H,*
3. *H is rigid.*

THEOREM 2.2. *For any two poitive integers k, l and for every graph G with $\chi(G) = k$, $G \not\supseteq K_k$, and with girth $G = l$ there exists a graph H with the following properties*:

1. $\chi(H) = k$,
2. *G is induced subgraph of H,*
3. *H is rigid,*
4. *girth $H = l$.*

In the proof we shall make use of the following:

LEMMA 2.3. *For every k (possibly infinite) and every odd l, $l \geqslant 3$ there exists a rigid graph H with at least k vertices, chromatic number 3 which contains every edge in an odd circuit of length l while it does not contain a circuit of length $< l$.*

Proof. This is a routine application of rigid-graph calculi. One starts from a rigid directed graph H' with k vertices (see [12]) and then replaces every edge of H' by a convenient rigid graph with large girth, e.g., the subdivision of K_4 indicated on Fig. 1. The resulting graph H has the desired properties; see [6, 10] for details. ∎

Proof of Theorem 2.1. Let G be a given graph as in Theorem 2.1. Let l' be the length of the shortest odd cycle in G. (This notation anticipates the proof of Theorem 2.2 which follows.)

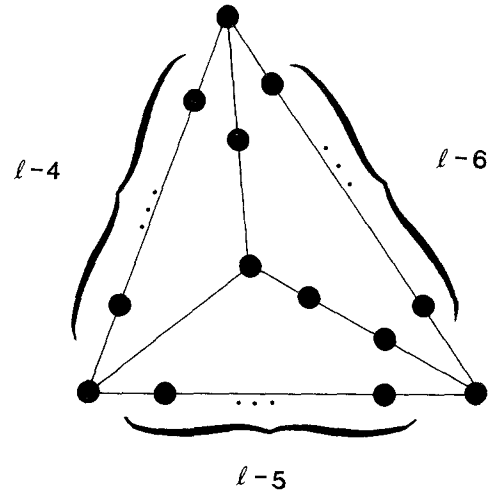

FIGURE 1

Without loss of generality we may assume that G is connected and every edge of G belongs to a circuit of length l'.

Let H_1 be a connected graph whose existence is guaranteed by Proposition 1.1: $\chi(H_1)=k$, H_1 does not contain odd circuits of length $<l'+4$ and there is no homomorphism $H_1 \to G$. Clearly we may assume that every edge of H_1 belongs to a circuit of length $l'+4$ and that $|V(H_1)| \geqslant |V(G)|$.

Let H_2 be a connected rigid graph guaranteed by Lemma 2.3. Let every edge of H_2 belong to an odd circuit of length $l'+2$ and let H_2 have girth $l'+2$. Also let $|V(H_2)| \geqslant |V(H_1)|$.

Let

$$\varphi_1\colon\ V(G) \to V(H_1)$$

and

$$\varphi_2\colon\ V(H_1) \to V(H_2)$$

be 1–1 mappings.

Let us construct the graph H as follows: First, take the disjoint union of graphs G, H_1, and H_2. Second, to every $v \in V(G)$ add a path of length $3l'$ from v to $\varphi_1(v)$; paths are supposed to be mutually disjoint and, apart from their endpoints, disjoint from $V(G)$ and $V(H_1)$.

Third, to every $v \in V(H_1)$ add a copy of the graph depicted in Fig. 2. Vertex 0 is identified with v and vertex $3l'$ is identified with $\varphi_2(v)$; paths are supposed to be again mutually disjoint. The construction is schematically depicted on Fig. 3.

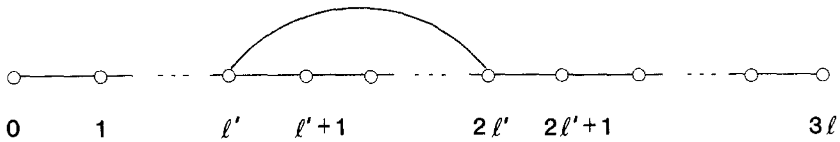

FIGURE 2

It is clear that $\chi(H)=\max\{\chi(G), \chi(H_1), \chi(H_2)\}=\chi(G)$. We prove that H is rigid.

Let f: $H \to H$ be a homomorphism. Clearly the homomorphic image of an odd circuit of length c contains an odd circuit of length $\leqslant c$. It follows from this and from the connectivity of graphs G, H_1, H_2 that $f(V(H_1))$ is a subset of either $V(G)$ or $V(H_1)$ or $V(H_2)$. Using the same argument $f(V(G)) \subseteq V(G)$ and thus it follows that $f(V(H_1)) \subseteq V(H_1)$.

From the circuit lengths further follows that $f(V(H_2)) \cap V(H_1) = \phi$. Suppose that $f(V(H_2)) \subseteq V(G)$. But then the paths between H_1 and H_2 have to be mapped to paths between G and H_1 (as $f(V(H_1)) \subseteqq V(H_1)$) which is impossible because of the lengths of the cycles.

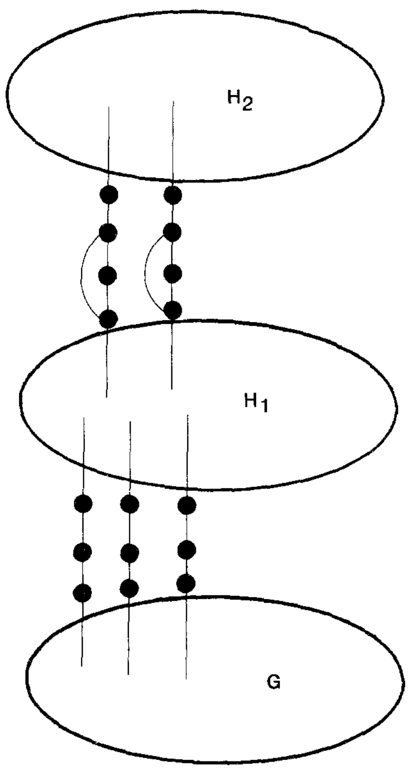

FIGURE 3

Consequently f induces homomorphisms $G \to G$, $H_1 \to H_1$, $H_2 \to H_2$. Thus $f \upharpoonright V(H_2)$ is an identity and this in turn yields identity $f \upharpoonright V(H_1)$ and finally $f \upharpoonright V(H_2)$. Thus $f(x) = x$ for every $x \in V(H)$. ∎

Proof of Theorem 2.2. The proof is similar to that of Theorem 2.1 and we shall stress only the differences.

Let l' be a least odd number $\geqslant l$. Let G be a given graph as in Theorem 2.1. Assume without loss of generality that G is connected and that every edge of G belongs to a circuit of length l'.

Let H_1 be a connected graph whose existence is guaranteed by Proposition 1.2: $\chi(H_1) = k$, H_1 has girth $l' + 4$, there is no homomorphism $H_1 \to G$. Clearly we may assume that every edge of H_1 belongs to a circuit of length $l' + 4$ and that $|V(H_1) \geqslant |V(G)|$.

Let H_2 be graphs as in the proof of Theorem 2.1. Starting from graphs G, H_1, H_2 we construct graph H as in the proof of Theorem 2.1.

One proves again that H is rigid. It is clear that the girth of H is equal to the girth of G. ∎

Remark. Put

$$\omega(G) = \sup\{k \mid K_k \subseteq G\}$$

and

$$g_o(G) = \min\{l \mid C_l \subseteq G \text{ and } l \text{ odd}\}.$$

The above proof of Theorem 2.1 implies in fact the following slightly stronger:

THEOREM 2.3. *For every graph G with $K_{\chi(G)} \nsubseteq G$ there exists a graph H with the following properties*:

1. $\chi(H) = \chi(G)$,
2. *G is an induced subgraph of H*,
3. $\omega(H) = \omega(G)$,
4. $g_o(H) = g_o(G)$,
5. *H is rigid.*

3. CONCLUDING REMARKS

1. A category of graphs $\mathscr{K}$ is called universal if the category of all graphs and all their homomorphisms can be embedded into $\mathscr{K}$; see, e.g., [11] for undefined notions.

Using standard techniques one can derive from Theorems 2.1 and 2.2

statements concerning universality of corresponding categories. Thus, e.g., Theorem 2.1 implies that the category $\mathscr{K}_G$ of all graphs H which contain a given graph G and which have the same chromatic number as G is universal if and only if every clique of G is smaller than the chromatic number of G.

2. In the infinite case we may consider the following question (motivated by Theorem 2.3):

Is it true that for every infinte graph G, $K_{\chi(G)} \nsubseteq G$, there exists a graph with the following properties:

$$\begin{aligned} &1.\quad \chi(G)=\chi(H),\\ &2.\quad \omega(G)=\omega(H),\\ &3.\quad g_o(G)=g_o(H),\\ &4.\quad |V(G)|=|V(H)|,\\ &5.\quad G \text{ is an induced subgraph of } H,\\ &6.\quad H \text{ is rigid.} \end{aligned} \tag{1}$$

One can easily extend the above proof of Theorem 2.1 and 2.2 to a proof of (1) when $\chi(G)$ is finite.

However, for infinite chromatic number $\chi(G)$ the answer to (1) is in general negative.

This is related to the following concept: let g be a class of graphs. A graph $H \in g$ is called *hom-universal* in g if for every graph $G \in g$ there exists a homomorphism $G \to H$.

Clearly if there exists a hom-universal graph in the class of all (say) countable graphs with given ω and g_0 then the answer to (1) is negative (as we may choose for G a hom-universal graph).

One may also prove that there exists a countable hom-universal graph for the class of all countable triangle-free graphs. (The same is true for countable K_k-free graphs.) Thus in this case, the answer to (1) is negative. We obtain similar results for classes of graphs which do not contain odd cycles of length $< l$.

On the other hand one can prove that there is no countable C_{2l}-free hom-universal graph and no K_ω-free countable hom-universal graph (in these proofs we apply methods of [9]).

This leads to the following particular case of (1):

Problem. Is it true that for every K_ω-free countable graph G there exists a countable rigid graph H which contains G?

Let us remark that related results concerning universal graphs in various classes of graphs were obtained by P. Komjath and J. Pach [7]; see also [13]. (However, the definition differs slightly from ours as they consider subgraphs instead of homomorphisms.)

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